Objectives:

- Solve for an unknown rate of change using related rates of change.

1. Draw a diagram.

2. Label your diagram, including units. If a quantity in the diagram is not changing, label it with a number. If a quantity in the diagram is changing, label it with a function.
   ex.) If a person is 6 feet tall, label the person's height as ___6ft___.
   If a person is running away from a tree, label distance between person and tree as ___$x(t)$ m___.

3. Identify the rates of change you know.
   ex.) Suppose the problem states that a buffalo's velocity after 3 hours of running is 40 miles per hour, and you labeled the buffalo's potion in the diagram as $x(t)$. Then we know the problem will involve the buffalo's velocity, ___$\frac{dx}{dt}$mph___. We also know that ___at t=3 hours, $\frac{dx}{dt} = 40$mph___
   ex.) If the problem states that a beam from the lighthouse is turning once per minute and you labeled the angle of the beam as θ in your diagram, we know the problem will involve ___$\frac{d\theta}{dt}$ radians per minute___ and ___$\frac{d\theta}{dt} = 2\pi$ radians per minute___.

4. Identify the rate of change that you wish to find.
   ex.) If the problem asks you to find how fast the height $h(t)$ of a rising balloon is increasing at $t = 3$, write ___$h'(3) =?$___ or ___$\frac{dh}{dt}$=? at t=3___

5. Write a function that describes the relationship between the quantities in the problem. Whenever possible, use creative methods to reduce the number of variables in the function.
   ex.) If we know $\frac{dr}{dt}$, the rate of change of the radius of a circle and want to know $\frac{dA}{dt}$, the rate of change of the area of the same circle, write ___$A = \pi r^2$___.

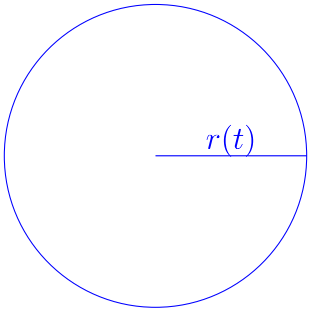
6. Take the derivative of the function you found in part 5 with respect to time.
   ex.) ___$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$___

7. Substitute actual numbers for every known quantity in the derivative you found in part 6. Then solve for the unknown rate of change. Remember to use units when stating your answer.
   We ONLY substitute ___after taking the derivative___.

8. Intuition check! Check to make sure that your understanding of the physical scenario makes sense with the answer you found.

1. Chocolate milk is spilling onto the floor and it accumulates in a circular puddle. The radius of the puddle increases at a rate of 4 cm/min. How fast is the area of the puddle increasing when the radius is 5 centimeters?

First, we draw a diagram and label it.



Information given: $\dfrac{dr}{dt} = 4$ cm/min for all values of t

Information sought: If A is the area of the puddle, we want to find $\dfrac{dA}{dt}$ when $r = 5$ cm

Next, write a function that relates area and radius.

$A = \pi r^2$ or $A(t) = \pi r(t)^2$ if we would really like to emphasize that each is a function of t

Differentiate the function *with respect to t* since we want the rate of change of area *with respect to time.*

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}$$

We want to solve for $\dfrac{dA}{dt}$ at $r = 5$, so we will plug in numbers for all other quantities in the derivative

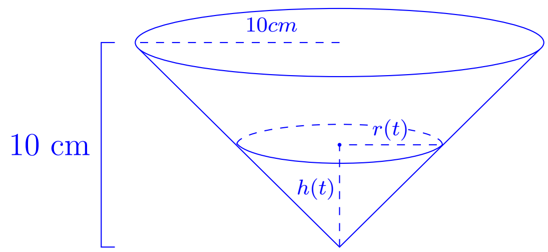
$$\frac{dA}{dt} = \pi \cdot 2(5) \cdot 4 = 40\pi$$

The puddle's area is increasing at $\boxed{40\pi \text{ cm}^2/\text{min}}$

Intuition check! The puddle's area should be increasing because more and more chocolate milk is spilling into the puddle, so $\dfrac{dA}{dt}$ should be positive. Indeed, we did find a positive value for $\dfrac{dA}{dt}$. Yay!

2. A conical paper cup is 10 cm tall with a radius of 10 cm. The cup is being filled with water so that the water level rises at a rate of 2 cm/sec. At what rate is water being poured into the cup when the water level is 8 centimeters?

First, we draw a diagram and label it accordingly.



Information given: $\frac{dh}{dt} = 2$ cm/sec

Information sought: If V is the volume of water in the cup, we seek $\frac{dV}{dt}$ when $h = 8$.

Next, we write a function that relates the volume and height of the water.

$$V = \frac{1}{3}\pi r^2 h$$

This function has 3 changing quantities. We can use our knowledge of similar triangles to write r in terms of h.

$$\frac{10}{10} = \frac{r}{h} \quad \Longrightarrow \quad h = r$$

Use $h = r$ to simplify the equation.

$$V = \frac{1}{3}\pi h^3$$

Differentiate with respect to t.

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot 3h^2 \frac{dh}{dt} = \pi h^2 \frac{dh}{dt}$$

Plug in values for every quantity except $\frac{dV}{dt}$.

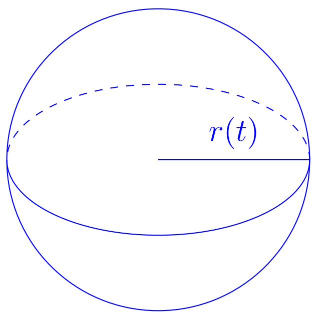
$$\frac{dV}{dt} = \pi 8^2 \cdot 2 = 128\pi$$

When the water is 8 cm high, the volume of water is increasing at $\boxed{128\pi \text{ cm}^3/\text{sec}}$.

Intuition check! The water is being poured into the cup, so the volume is increasing. This means $\frac{dV}{dt}$ should always be positive. Indeed, we found a positive value for $\frac{dV}{dt}$. Whoopee!

3. A spherical balloon is inflated so that r, its radius, increases at a rate of $\frac{2}{r}$ cm/sec. How fast is the volume of the balloon increasing when the radius is 4 centimeters?

First, we draw a diagram and label it accordingly.



Information given: $\frac{dr}{dt} = \frac{2}{r}$ for all t

Information sought:
If V is the volume of the balloon, we seek $\frac{dV}{dt}$ at $r = 4$ cm.

Next, we write a function that relates the volume and and radius of the balloon.

$$V = \frac{4}{3}\pi r^3$$

Take the derivative with respect to t.

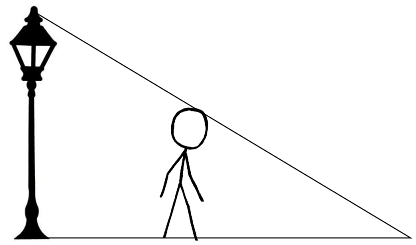
$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

Plug in numbers for all quantities except for the one we seek, which is $\frac{dV}{dt}$.

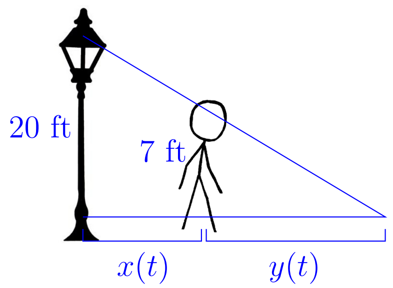
$$\frac{dV}{dt} = \frac{4}{3}\pi 3(4)^2 \frac{2}{r} = 32\pi$$

So when the radius is 4 cm, the volume is growing at $\boxed{32\pi \text{ cm}^3/\text{sec}}$.

4. A 7 ft tall person is walking away from a 20 ft tall lamppost at a rate of 5 ft/sec. At what rate is the length of the person's shadow changing when the person is 16 feet from the lamppost?



First, we draw a diagram and label it accordingly.



Information given: $\frac{dx}{dt} = 5$ ft/sec for all t.

Information sought: $\frac{dy}{dt}$ at $x = 16$ ft.

Next, we write a function that relates the the quantities in the diagram. Notice that we can use our the fact that similar triangles have proportional side lengths. We will also simplify the equation a bit to make it easier to work with.

$$\frac{7}{y} = \frac{20}{x+y} \qquad \Longrightarrow \qquad 7(x+y) = 20y \qquad \Longrightarrow \qquad 7x = 13y$$

Now differentiate with respect to t.

$$7\frac{dx}{dt} = 13\frac{dy}{dt}$$

Plug in values for every quantity except $\frac{dy}{dt}$. Solve.

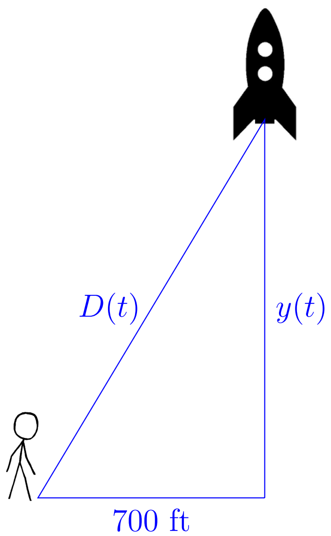
$$7 \cdot 5 = 13\frac{dy}{dt} \qquad \Longrightarrow \qquad \frac{dy}{dt} = \frac{35}{13}$$

(Notice that we did need to plus in $x = 16$ anywhere. This happens sometimes.)
When the person is 16 feet from the lamppost, the shadow's length is growing at a rate of $\boxed{\frac{35}{13} \text{ ft/sec}}$.

Intuition check! As the person walks away, the shadow should get longer, which means y is increasing. It makes sense that the value of $\frac{dy}{dt}$ should be positive. Wowee!

5. An observer stands 700 feet away from a launch pad to observe a rocket launch. The rocket blasts off and maintains a velocity of 900 ft/sec. How fast is the observer-to-rocket distance changing when the rocket is 2400 feet from the ground?

First, we draw a diagram and label it accordingly.



Information given: $\frac{dy}{dt} = 900$ ft/sec for all t.

Information sought: $\frac{dD}{dt}$ at $y = 2400$ ft.

Next, we write a function that relates the the quantities in the diagram. Use the Pythagorean Theorem!

$$700^2 + y^2 = D^2$$

Differentiate with respect to t.

$$2y\frac{dy}{dt} = 2D\frac{dD}{dt}$$

Now we wish to plug in specific numbers for every quantity in the above equation except for $\frac{dD}{dt}$. However, we notice that we don't have a specific value for D at $y = 2400$. So first we need to find D at $y = 2400$ using the Pythagorean Theorem.

$$700^2 + 2400^2 = D^2 \qquad \Longrightarrow \qquad D = 2500$$

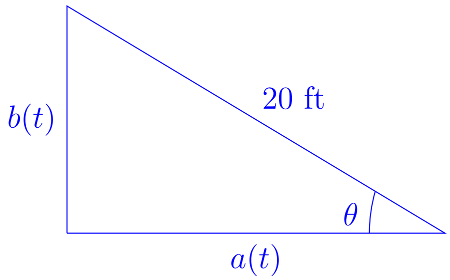
At this point we can finish the problem by plugging in numbers for every quantity in the equation containing $\frac{dD}{dt}$.

$$2 \cdot 2400 \cdot 900 = 2 \cdot 2500\frac{dD}{dt} \qquad \Longrightarrow \qquad D = 864$$

When the rocket is 2400 feet from the ground, the distance between the observer and the rocket is increasing at a rate of $\boxed{864 \text{ ft/sec}}$.

Intuition check! As the rocket goes up, the observer-to-rocket distance should be increasing and $\frac{dD}{dt}$ should be positive, which it is. Stupendous!

6. A 20ft ladder is left leaning against the wall and begins to slide down the wall. As the ladder slides, the angle between the ladder and the ground is decreasing by 5 radians per second. Find the rate at which the top of the ladder is moving down the wall when the top of the ladder hits the ground.



First, we draw and label the diagram.

Our known rate of change is $\frac{d\theta}{dt} = -5$. The rate of change we want to find is $\frac{db}{dt}$ when $b(t) = 0$, which is the same as when $\theta = 0$.

Next we write an equation containing θ and b:

$$\sin(\theta) = \frac{b}{20}$$

Then we differentiate with respect to t :

$$\cos(\theta)\frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{db}{dt}$$

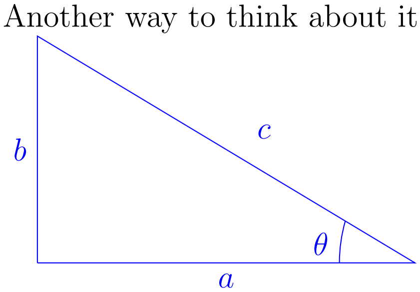
Assuming $b(t) = 0$ and $\theta = 0$, we substitute and solve:

$$\cos(0)(-5) = \frac{1}{20} \cdot \frac{db}{dt}$$

$$1(-100) = \frac{db}{dt}$$

So, when the top of the ladder hits the ground, it is moving at $100\frac{100\text{ ft}}{\text{sec}}$.

Another way to think about it: Set up the equation without plugging in any constants.



$\sin(\theta) = \frac{b}{c} = bc^{-1}$ Then, when we differentiate, we get $\cos(\theta)\frac{d\theta}{dt} = b(-c^{-2})\frac{dc}{dt} + (1)c^{-1}\frac{db}{dt}$. Then we want to plug in everything we know is true when the ladder hits the ground. We know $\theta = 0$. We also know $c = 20$ and $\frac{d\theta}{dt} = -5$, since this is always true. The last important thing to notice is that c, the length of the ladder, doesn't change throughout the problem, so $\frac{dc}{dt} = 0$. Then, subbing in, we get $\cos(0) = b(c^{-2})(0) + 20^{-1}\frac{db}{dt} = \frac{1}{20} \cdot \frac{db}{dt}$ just like the other method.